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The "geometric buckling" of spheroidal nuclear reactors

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SUMMARY

The Helmholtz equation $\nabla^2\phi + B^2\phi = 0$ has been solved with the boundary condition $\phi = 0$ on the surface of a spheroid. The lowest eigenvalue has been calculated for prolate and oblate spheroids of different shapes, and comprehensive tables of the "geometric buckling" of spheroidal nuclear reactors are given.

Introduction

The partial differential equation

$$\nabla^2\phi + B^2\phi = 0,$$

the Helmholtz equation, arises often in mathematical physics, for instance in connection with problems on diffusion and heat conduction [1]. To take a specific example, the equation is valid for the neutron flux ϕ in a critical nuclear reactor according to the one group diffusion theory [2]. The lowest eigenvalue with the condition that the flux vanishes at the effective boundaries of the reactor is called the "geometric buckling" B_g^2 of the reactor and is related to its critical size.

It has been shown that it is possible to separate the above equation in eleven orthogonal coordinate systems [1, 3], and the resulting ordinary differential equations have also been listed [4]. Most of these equations do not lead to known functions, however. Only in four coordinate systems is it possible to obtain functions for which sufficiently accurate tabulated values have been available. These systems include the three familiar ones—Cartesian, spherical, and circular cylindric—and the elliptic cylindric coordinate system has also been treated thoroughly [5].

As extensive tables of the spheroidal functions have been published recently by Stratton *et al.* [6], it has now become possible to treat two more cases numerically, namely, the prolate and oblate spheroidal systems. In the following it will be shown how the geometric buckling can be calculated for reactors of spheroidal shapes. The notation is the same as that used by Morse and Feshbach [1], and in the tables by Stratton *et al.* [6].

Prolate spheroid

We introduce the coordinates

$$\begin{aligned} x &= a\sqrt{\xi^2-1}\,\sqrt{1-\eta^2}\,\cos\varphi & 1 &\leqslant \xi < \infty \\ y &= a\sqrt{\xi^2-1}\,\sqrt{1-\eta^2}\,\sin\varphi & -1 &\leqslant \eta \leqslant 1 \\ z &= a\,\xi\,\eta & 0 &\leqslant \varphi \leqslant 2\pi. \end{aligned}$$

By elimination of η and φ we obtain

$$\frac{x^2+y^2}{a^2(\xi^2-1)} + \frac{z^2}{a^2\xi^2} = 1$$

which is the equation of a prolate spheroid with semi-major axis $M_1 = a\xi$ and semi-minor axis $M_2 = a\sqrt{\xi^2-1}$. The foci are at the points $z = \pm a$ and the eccentricity $e = 1/\xi$.

In the above coordinates the Helmholtz equation is

$$\frac{\partial}{\partial\xi}\left[(\xi^2-1)\frac{\partial\phi}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\phi}{\partial\eta}\right] + \frac{\xi^2-\eta^2}{(\xi^2-1)(1-\eta^2)}\frac{\partial^2\phi}{\partial\varphi^2} + B^2a^2(\xi^2-\eta^2)\,\phi = 0.$$

By putting

$$\phi(\xi,\eta,\varphi) = J(\xi)\,S(\eta)\,F(\varphi)$$

we can separate this equation into

$$\frac{d}{d\xi}\left[(\xi^2-1)\frac{dJ}{d\xi}\right] - \left(A - h^2\xi^2 + \frac{m^2}{\xi^2-1}\right)J = 0$$

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dS}{d\eta}\right] + \left(A - h^2\eta^2 - \frac{m^2}{1-\eta^2}\right)S = 0$$

$$\frac{d^2F}{d\varphi^2} + m^2F = 0$$

where $h = Ba$, and m and A are separation constants.

The condition that the solution must be finite at $\eta = \pm 1$ implies that the separation constant A can assume only certain discrete values for given m and h.

The solutions to these equations are respectively

$$J(\xi) = je_{ml}(h,\xi) = \left(\frac{\xi^2-1}{\xi^2}\right)^{\frac{m}{2}} \sum_n{}' \, a_n(h/ml)\, j_{n+m}(h\xi)$$

$$S_{ml}(h,\eta) = \sum_n{}' \, d_n(h/ml)\, P^m_{m+n}(\eta)$$

$$F(\varphi) = C_1\cos m\varphi + C_2\sin m\varphi$$

where P^m_{m+n} denotes associated Legendre functions and j_{n+m} spherical Bessel functions defined by

$$j_k(x) = \sqrt{\frac{\pi}{2x}}\, J_{k+1/2}(x).$$

The summations should be done over either even or odd n-values, and the expansion coefficients a_n and d_n are tabulated in the tables by Stratton *et al.* [6] for different m-, l- and h-values.

In our special case we want a solution that is independent of φ, and hence $m=0$. The requirement of the lowest possible eigenvalue leads, in analogy with the spherical case, to that $l=0$. The boundary condition that the solution should vanish at the surface of a prolate spheroid determined by $\xi=\xi_0$ implies that $x_0=h\,\xi_0$ should be the first zero of the function

$$j\,e_{00}(h,\xi) = \sum_n{}' \, a_n(h/00)\, j_n(h\,\xi).$$

Using tables of the spherical Bessel functions [7], these first zeros have been evaluated for $h=0\,(0.1)\,1$ and for $h=1\,(0.2)\,8$, for which values the expansion coefficients are tabulated [6]. From these zeros the eccentricity $e=1/\xi_0$ and the axis ratio $c=\frac{\xi_0}{\sqrt{\xi_0^2-1}}$ are easily calculated, as is the buckling times the square of the semi-minor axis $B_g^2 M_2^2$, which is equal to $x_0^2-h^2$. These values are given in table 1. It is believed that the values for x_0 are correct within two units in the last figure.

In the limiting case of a sphere, that is for $e=0$, we find a buckling of $B_g^2=\frac{\pi^2}{M_2^2}$ as expected. We also see that for $e\rightarrow 1$ we approach the shape of an infinite circular cylinder and $B_g^2\rightarrow(2.405)^2/M_2^2$.

The following approximate formula gives the buckling within 1 per cent for $e<0.8$

$$B_g^2=\frac{\pi^2}{M_2^2}\left(1-\frac{e^2}{3}\right).$$

Oblate spheroid

In this case we introduce the coordinates

$$\begin{aligned} x &= b\sqrt{\xi^2+1}\,\sqrt{1-\eta^2}\,\cos\varphi & 0 &\leqslant \xi < \infty \\ y &= b\sqrt{\xi^2+1}\,\sqrt{1-\eta^2}\,\sin\varphi & -1 &\leqslant \eta \leqslant 1 \\ z &= b\,\xi\,\eta & 0 &\leqslant \varphi \leqslant 2\pi. \end{aligned}$$

By elimination of η and φ we get

$$\frac{x^2+y^2}{b^2(\xi^2+1)}+\frac{z^2}{b^2\xi^2}=1$$

which is the equation of an oblate spheroid with semi-major axis $M_1=b\sqrt{\xi^2+1}$

Table 1. Prolate spheroids.

Values of x_0, eccentricity, axis ratio, and buckling for h-values from 0 to 8.

h	x_0	$e = \frac{1}{\xi_0}$	$c = \frac{M_1}{M_2}$	$B_g^2 M_2^2$
0	3.14159	0	1.0000	9.8696
0.1	3.14265	0.031820	1.0005	9.8662
0.2	3.14584	0.063576	1.0020	9.8563
0.3	3.15116	0.095203	1.0046	9.8398
0.4	3.15861	0.126638	1.0081	9.8168
0.5	3.16822	0.157817	1.0127	9.7876
0.6	3.17999	0.188680	1.0183	9.7523
0.7	3.19396	0.219164	1.0249	9.7114
0.8	3.21013	0.249211	1.0326	9.6649
0.9	3.22852	0.278766	1.0413	9.6133
1.0	3.24917	0.307771	1.0510	9.5571
1.2	3.29729	0.363935	1.0736	9.4321
1.4	3.35462	0.417335	1.1004	9.2935
1.6	3.42125	0.467665	1.1313	9.1449
1.8	3.49718	0.514700	1.1664	8.9903
2.0	3.58231	0.558299	1.2053	8.8329
2.2	3.67642	0.598408	1.2481	8.6761
2.4	3.77919	0.635057	1.2946	8.5222
2.6	3.89021	0.668344	1.3444	8.3736
2.8	4.00897	0.698434	1.3973	8.2319
3.0	4.13493	0.725526	1.4531	8.0977
3.2	4.26752	0.749850	1.5115	7.9717
3.4	4.40615	0.771649	1.5722	7.8541
3.6	4.55027	0.791162	1.6351	7.7449
3.8	4.69933	0.808626	1.6997	7.6437
4.0	4.85285	0.824258	1.7661	7.5501
4.2	5.01036	0.838260	1.8340	7.4637
4.4	5.17145	0.850825	1.9031	7.3839
4.6	5.33574	0.862111	1.9735	7.3101
4.8	5.50290	0.872267	2.0449	7.2419
5.0	5.67264	0.881424	2.1172	7.1788
5.2	5.84469	0.889696	2.1903	7.1204
5.4	6.01882	0.897186	2.2642	7.0662
5.6	6.19482	0.903981	2.3388	7.0158
5.8	6.37251	0.910159	2.4140	6.9689
6.0	1.55173	0.915789	2.4896	6.9252
6.2	6.73234	0.920928	2.5659	6.8844
6.4	6.91420	0.925631	2.6426	6.8462
6.6	7.09721	0.929943	2.7197	6.8104
6.8	7.28127	0.933903	2.7969	6.7769
7.0	7.46628	0.937549	2.8747	6.7453
7.2	7.65216	0.940911	2.9529	6.7156
7.4	7.83886	0.944015	3.0310	6.6877
7.6	8.02627	0.946891	3.1098	6.6610
7.8	8.21440	0.949552	3.1887	6.6364
8.0	8.40313	0.952026	3.2679	6.6126
∞	∞	1.0000	∞	5.7831

Table 2. Oblate spheriods.

Values of x_0, eccentricity, axis ratio, and buckling for g-values from 0 to 8.

g	x_0	$e = \frac{1}{\sqrt{\xi_0^2+1}}$	$c = \frac{M_1}{M_2}$	$B_g^2 M_2^2$
0	3.14159	0	1.0000	9.8696
0.1	3.14053	0.031826	1.0005	9.8629
0.2	3.13735	0.063619	1.0020	9.8430
0.3	3.13206	0.095347	1.0046	9.8098
0.4	3.12466	0.126978	1.0082	9.7635
0.5	3.11517	0.158477	1.0128	9.7043
0.6	3.10362	0.189808	1.0185	9.6325
0.7	3.09002	0.220938	1.0253	9.5482
0.8	3.07440	0.251828	1.0333	9.4519
0.9	3.05682	0.282436	1.0424	9.3442
1.0	3.03730	0.312726	1.0528	9.2252
1.2	2.99272	0.372169	1.0774	8.9564
1.4	2.94122	0.429789	1.1075	8.6508
1.6	2.88358	0.485183	1.1436	8.3150
1.8	2.82077	0.537932	1.1863	7.9567
2.0	2.75402	0.587610	1.2359	7.5846
2.2	3.68475	0.633822	1.2929	7.2079
2.4	2.61452	0.676238	1.3574	6.8357
2.6	2.54493	0.714633	1.4296	6.4767
2.8	2.47744	0.748929	1.5091	6.1377
3.0	2.41324	0.779186	1.5954	5.8237
3.2	2.35321	0.805620	1.6880	5.5376
3.4	2.29785	0.828528	1.7859	5.2801
3.6	2.24732	0.848284	1.8884	5.0505
3.8	2.20154	0.865276	1.9948	4.8468
4.0	2.16026	0.879881	2.1044	4.6667
4.2	2.12312	0.892453	2.2166	4.5076
4.4	2.08971	0.903301	2.3310	4.3669
4.6	2.05965	0.912689	2.4470	4.2422
4.8	2.03254	0.920845	2.5646	4.1312
5.0	2.00805	0.927961	2.6833	4.0323
5.2	1.98585	0.934194	2.8030	3.9436
5.4	1.96568	0.939679	2.9235	3.8639
5.6	1.94728	0.944525	3.0447	3.7919
5.8	1.93046	0.948824	3.1665	3.7267
6.0	1.91503	0.952653	3.2888	3.6673
6.2	1.90084	0.956075	3.4116	3.6132
6.4	1.88774	0.959147	3.5347	3.5636
6.6	1.87561	0.961912	3.6582	3.5179
6.8	1.86437	0.964409	3.7819	3.4759
7.0	1.85391	0.966672	3.9060	3.4370
7.2	1.84416	0.968728	4.0303	3.4009
7.4	1.83506	0.970601	4.1547	3.3674
7.6	1.82653	0.972313	4.2794	3.3362
7.8	1.81853	0.973881	4.4042	3.3071
8.0	1.81101	0.975321	4.5292	3.2798
∞	1.57080	1.0000	∞	2.4674

and semi-minor axis $M_2 = b\,\xi$. The foci are at $z = \pm b$, and the eccentricity $e = 1/\sqrt{\xi^2+1}$.

If we write $g = Bb$, the solution can be obtained from the prolate case by the substitutions [6]

$$\begin{aligned} \xi &\rightarrow -i\,\xi \\ a &\rightarrow \ \ i\,b \\ h &\rightarrow \ \ i\,g. \end{aligned}$$

Proceeding in the same way as in the prolate case, but for an oblate spheroid given by $\xi = \xi_0$, we have to determine the first zero of the function

$$j\,e_{00}\,(i\,g,\ -i\,\xi) = \sum_n{}' \, a_n\,(i\,g/00)\,j_n\,(g\,\xi)$$

for different g-values. The zeros have been evaluated for $g = 0\,(0.1)\,1$ and for $g = 1\,(0.2)\,8$. As before the eccentricity, the axis ratio, and the buckling times the square of the semi-minor axis $B_g^2\,M_2^2$, which is equal to x_0^2, have also been calculated, and are given in table 2. The values for x_0 are believed to be correct within one unit in the last figure.

Also in this case $e = 0$ gives the correct buckling of a sphere. As $e \rightarrow 1$ we approach an infinite slab and $B_g^2 \rightarrow \frac{\pi^2}{4\,M_2^2}$. The approximate formula

$$B_g^2 = \frac{\pi^2}{M_2^2}\left(1 - \frac{2\,e^2}{3}\right)$$

gives the buckling within 1 per cent for $e < 0.8$.

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